On the numerical solution of fuzzy elliptic PDEs by means of polynomial response surfaces

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Abstract

Physical models often have some uncertainty in their parameters. This uncertainty is typically modeled by means of random numbers, random fields or random processes. Results from calculations with these random parameters are however not very reliable if the probability distribution is not well known. Fuzzy numbers offer an alternative and can be used very effectively as a kind of reliability or worst-case analysis. In this paper, we consider the solution of elliptic partial differential equations with a fuzzy diffusion coefficient. Inspired by popular solution techniques for stochastic elliptic PDEs, we make use of orthogonal sets of polynomials and a Galerkin projection to construct a response surface. This approximation converges very rapidly to the exact solution and is shown to be much more accurate than response surfaces constructed by alternative methods, such as Kriging, at a comparable cost in computation time.

1 Introduction

Fuzzy variables and fields offer an alternative to random variables and fields for modeling uncertainty in physical models. They can be used in early design stages when little is known about the details of the design. Also, they are often preferred to random variables if the probability distribution of the uncertain parameters is not well known [14]. In this paper, we consider the diffusion equation, with an uncertain diffusion coefficient. This coefficient is modeled as a fuzzy field, i.e., it is a fuzzy number that varies over the spatial domain.

Solving a fuzzy PDE can be very costly. Calculating the fuzzy solution in one point in the domain by the so-called α-cut approach comes down to minimizing and maximizing the solution value over the parameter domain determined by different α-cuts of the fuzzy input parameters. In order to speed up this optimization process, the technique of response surfaces can be used [9, 10, 15]. This was done, e.g., in [6, 7, 12, 1].

Because fuzzy numbers are often used in a kind of reliability or worst-case analysis, it is important that the numerical solution of the fuzzy equation is accurate. Black-box approaches, like Kriging, can be applied to construct the response surfaces to approximate the exact solution surface. For certain problems one could do better than Kriging by choosing a more appropriate type of response surface that incorporates some knowledge about the properties of the model. Elliptic PDEs for example can be proven to be analytic as a function of their diffusion coefficient under some mild conditions [5].

Hence, the solution surface is very smooth and can be very well approximated by polynomials. Response surfaces constructed by polynomials chaos expansions are a popular technology when solving stochastic PDEs [3, 2, 13, 18]. We apply such expansions here to solve a fuzzy elliptic PDE. We show through numerical experiments that they are more accurate and have a faster and more regular convergence than response surfaces constructed by a Kriging approach.

2 Problem Description

We consider the fuzzy elliptic PDE on a d-dimensional Lipschitz domain Ω:

$$\begin{aligned} -\nabla\cdot\left(\widetilde{a}(\mathbf{x})\nabla\widetilde{u}(\mathbf{x})\right) &= f(\mathbf{x}) \text{ in } \Omega\subset\mathbb{R}^d, \\ \widetilde{u}(\mathbf{x}) &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{1}$$

with $\widetilde{u}$ the fuzzy unknown, f the source term and $\mathbf{n}$ the outward normal on $\partial\Omega$. In this paper we use the notation $\widetilde{u}$ to denote a fuzzy number or field, dependent on the context. The notation $\widetilde{u}(\cdot)$ is used to explicitly denote a fuzzy field, $\widetilde{u}(\mathbf{x})$ is a fuzzy field evaluated in $\mathbf{x}$ and finally $[\widetilde{u}]_\alpha$ and $[\widetilde{u}(\cdot)]_\alpha$ are used for the α-cut of $\widetilde{u}$ and $\widetilde{u}(\cdot)$ respectively. We refer to [11] for an introduction on fuzzy numbers.

The diffusion coefficient $\widetilde{a}$ is a fuzzy field that can be represented by a finite linear combination of basis functions $\{a_r(\mathbf{x})\}_{r=0}^{N_\xi}$ multiplied by non-interactive[1] fuzzy coefficients $\{\widetilde{\xi}_r\}_{r=0}^{N_\xi}$. More precisely, it is assumed that the diffusion coefficient satisfies:

$$\widetilde{a}(\mathbf{x}) = a_0(\mathbf{x}) + \sum_{r=1}^{N_\xi}\widetilde{\xi}_r a_r(\mathbf{x}), \tag{2}$$

with $[\widetilde{\xi}_r]_0 = [-1,1]$.

Problem (1) is typically solved by means of the α-cut approach. To that end, one first parameterizes the fuzzy PDE, i.e., it is rewritten as a regular PDE in the unknown $u(\mathbf{x},\boldsymbol{\xi})$:

$$\begin{aligned} -\nabla_x\cdot\left(a(\mathbf{x},\boldsymbol{\xi})\nabla_x u(\mathbf{x},\boldsymbol{\xi})\right) &= f(\mathbf{x},\boldsymbol{\xi}) \text{ in } \Omega, \\ u(\mathbf{x},\boldsymbol{\xi}) &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{3}$$

Here, $\boldsymbol{\xi}$ is a parameter that lies in the N_ξ-dimensional hypercube $[\widetilde{\boldsymbol{\xi}}]_0$. We will further on refer to this hypercube as the uncertainty domain and call it Ξ. Note that with every $\boldsymbol{\xi}\in\Xi$, there corresponds a PDE solution $u(\cdot,\boldsymbol{\xi})$. This mapping will be denoted by $S(\boldsymbol{\xi})$.

The fuzzy solution $\widetilde{u}(\cdot)\equiv u(\cdot,\widetilde{\boldsymbol{\xi}})$ is then defined by its α-cuts:

$$[\widetilde{u}(\cdot)]_\alpha = \{u(\cdot,\boldsymbol{\xi}) = S(\boldsymbol{\xi})\colon \boldsymbol{\xi}\in[\widetilde{\boldsymbol{\xi}}]_\alpha\}. \tag{4}$$

In fact, this α-cut approach defines how to extend a map between deterministic variables to a map between fuzzy variables and is commonly referred to as the so-called fuzzification of that map.

If one considers this fuzzy field $\widetilde{u}(\cdot)$ in one particular point $\mathbf{x}_i\in\Omega$ and assumes that S is continuous in $\boldsymbol{\xi}$, the α-cut approach comes down to solving the optimization problem

$$\forall\alpha\in[0,1]\colon [\widetilde{u}(\mathbf{x}_i)]_\alpha = \left[\min_{\boldsymbol{\xi}\in[\boldsymbol{\xi}]_\alpha} u(\mathbf{x}_i,\boldsymbol{\xi}), \max_{\boldsymbol{\xi}\in[\boldsymbol{\xi}]_\alpha} u(\mathbf{x}_i,\boldsymbol{\xi})\right]$$

subject to

$$u(\cdot,\boldsymbol{\xi}) = S(\boldsymbol{\xi}). \tag{5}$$

It is common to do this for a set of α-levels $0=\alpha_0<\alpha_1<\cdots<\alpha_{N_\alpha}=1$ and a set of points $\{\mathbf{x}_i\}_{i=1}^{N_x}$. Because of the PDE constraint (5), this can become very expensive. Response surfaces are therefore often used to replace the expensive operator S by a cheaper (but approximate) variant S^{r}. A typical black-box approach to construct such a surface, samples the parameterized equation in a set of points $\{\boldsymbol{\xi}_k\}_{k=1}^{N_s}$ in the

[1] non-interactiveness is the fuzzy equivalent of stochastic independence

parameter space Ξ. In addition, also a set of basis functions $\{r_j(\boldsymbol{\xi})\}_{j=1}^{N_r}$ is chosen. By linearly combining these functions $r_j(\boldsymbol{\xi})$, an interpolating surface through the solutions $\{S(\boldsymbol{\xi}_k)\}_{k=1}^{N_s}$ is then constructed as

$$u^{\mathrm{r}}(\mathbf{x},\boldsymbol{\xi}) = \sum_{j=1}^{N_{\mathrm{r}}} u_j^{\mathrm{r}}(\mathbf{x}) r_j(\boldsymbol{\xi}), \tag{6}$$

with the coefficients $u_j^{\mathrm{r}}(\mathbf{x})$. This response surface then defines the approximate solution operator by $S^{\mathrm{r}}(\boldsymbol{\xi}) \equiv u^{\mathrm{r}}(\cdot,\boldsymbol{\xi})$.

A problem with this approach, if used as a black-box, is that it does not guarantee accuracy or convergence to the exact solution. In Section 4, we will discuss some properties of the solution operator S and argue that a polynomial response surface is expected to be accurate and converge rapidly to the exact solution. But first, we need to define what accuracy in the fuzzy sense exactly means.

3 A distance measure

The distance function that we will use to measure the accuracy of an approximate solution of (1) is the supremum distance.

Definition 1. *The supremum distance d_∞ between the fuzzy variables $\widetilde{u}$ and $\widetilde{v}$ is defined as*

$$d_\infty(\widetilde{u},\widetilde{v}) = \sup_{0\leqslant\alpha\leqslant 1} d_H\left([\widetilde{u}]_\alpha,[\widetilde{v}]_\alpha\right)$$

with $d_H\left([\widetilde{u}]_\alpha,[\widetilde{v}]_\alpha\right)$ the Hausdorff distance:

$$d_H\left([\widetilde{u}]_\alpha,[\widetilde{v}]_\alpha\right) = \max\left\{\sup_{u\in[\widetilde{u}]_\alpha}\inf_{v\in[\widetilde{v}]_\alpha} d(u,v),\ \sup_{v\in[\widetilde{v}]_\alpha}\inf_{u\in[\widetilde{u}]_\alpha} d(u,v)\right\}.$$

For a $\widetilde{u}(\cdot)$ and $\widetilde{v}(\cdot)$, which are defined by fuzzification of $u(\cdot,\boldsymbol{\xi})$ and $v(\cdot,\boldsymbol{\xi})$ respectively, the Hausdorff distance $d_{\mathrm{H}}\left([\widetilde{u}]_\alpha,[\widetilde{v}]_\alpha\right)$ is equal to

$$\max\left\{\sup_{\boldsymbol{\xi}_1\in[\widetilde{\boldsymbol{\xi}}]_\alpha}\inf_{\boldsymbol{\xi}_2\in[\widetilde{\boldsymbol{\xi}}]_\alpha} d(u(\cdot,\boldsymbol{\xi}_1),v(\cdot,\boldsymbol{\xi}_2)),\ \sup_{\boldsymbol{\xi}_2\in[\widetilde{\boldsymbol{\xi}}]_\alpha}\inf_{\boldsymbol{\xi}_1\in[\widetilde{\boldsymbol{\xi}}]_\alpha} d(u(\cdot,\boldsymbol{\xi}_1),v(\cdot,\boldsymbol{\xi}_2))\right\}. \tag{7}$$

It is clear that this is not cheap, nor easy to compute. Therefore, we will prove the following proposition that gives us a cheap upper bound for the distance between two fuzzy sets and that we will use to estimate the accuracy of the approximate fuzzy solution of (1) defined by a response surface.

Proposition 3.1. *The supremum distance d_∞ between two fuzzy fields $\widetilde{u}(\cdot)\equiv u(\cdot,\widetilde{\boldsymbol{\xi}})$ and $\widetilde{v}(\cdot)\equiv u(\cdot,\widetilde{\boldsymbol{\xi}})$ is bounded from above as*

$$d_\infty(\widetilde{u},\widetilde{v}) \leqslant \sup_{\boldsymbol{\xi}\in\Xi} d(u(\cdot,\boldsymbol{\xi}),v(\cdot,\boldsymbol{\xi})).$$

Proof. The Hausdorff distance $d_{\mathrm{H}}\left([\widetilde{u}]_\alpha,[\widetilde{v}]_\alpha\right)$, given in (7), is smaller than $\sup_{\boldsymbol{\xi}\in[\widetilde{\boldsymbol{\xi}}]_\alpha} d(u(\cdot,\boldsymbol{\xi}),v(\cdot,\boldsymbol{\xi}))$ and therefore

$$d_{\mathrm{H}}\left([\widetilde{u}]_\alpha,[\widetilde{v}]_\alpha\right) \leqslant \sup_{\boldsymbol{\xi}\in\Xi} d(u(\cdot,\boldsymbol{\xi}),v(\cdot,\boldsymbol{\xi})).$$

Hence

$$d_\infty(\widetilde{u},\widetilde{v}) \equiv \sup_{0\leqslant\alpha\leqslant 1} d_{\mathrm{H}}\left([\widetilde{u}]_\alpha,[\widetilde{v}]_\alpha\right) \leqslant \sup_{\boldsymbol{\xi}\in\Xi} d(u(\cdot,\boldsymbol{\xi}),v(\cdot,\boldsymbol{\xi})).$$

$\square$

4 Polynomial Response Surface

To define the mathematical setting for our analysis, we rewrite problem (3) in its weak form and choose appropriate function spaces for the different coefficients. We take $a \in L^\infty(\Omega) \otimes L^\infty(\Xi)$ and $f \in H^{-1}(\Omega) \otimes L^2_w(\Xi)$, where the usual notation $L^2_w(\Xi)$ is used to denote the weighted L^2-space over the domain Ξ with weighting function $w(\boldsymbol{\xi})$. The weak formulation of (3) is then given by

Weak Formulation 1. *Find $u \in H^1_0(\Omega) \otimes L^2_w(\Xi)$ such that*

$$\int_\Xi \int_\Omega a \nabla_x u \cdot \nabla_x v \; w(\boldsymbol{\xi}) \mathrm{d}\mathbf{x} \mathrm{d}\boldsymbol{\xi} = \int_\Xi \int_\Omega f v \; w(\boldsymbol{\xi}) \mathrm{d}\mathbf{x} \mathrm{d}\boldsymbol{\xi}, \quad \forall v \in H^1_0(\Omega) \otimes L^2_w(\Xi).$$

Under the assumption of (2) and

$$0 < r \leqslant a(\mathbf{x}, \boldsymbol{\xi}) \leqslant R < \infty, \quad \mathbf{x} \in \Omega, \; \boldsymbol{\xi} \in \Xi, \tag{8}$$

for some $r, R \in (0, \infty)$, it is proven in [5] that $\boldsymbol{\xi} \to u(\cdot, \boldsymbol{\xi})$ is an analytic function over the domain Ξ.

Analytic functions can be approximated well by polynomials. A particularly convenient choice of basis for constructing the response surface is a set of orthonormal polynomials $\{\psi_j(\boldsymbol{\xi})\}_{j=1}^{N_\psi}$. The results from [3, 2, 5] show (near-)exponential convergence of the error

$$\|u - u^{\mathrm{F}}\|_{L^\infty(\Xi; H^1(\Omega))} \tag{9}$$

for the set of Legendre polynomials, as well as the set of Chebyshev polynomials, which are defined w.r.t. to the weighting functions $w(\boldsymbol{\xi}) = 1$ and $w(\boldsymbol{\xi}) = 1 / \prod_{r=1}^{N_\xi} \sqrt{1 - \xi_r^2}$ respectively. By the term u^{F}, we mean the truncated Fourier series

$$u^{\mathrm{F}}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{j=1}^{N_\psi} u^{\mathrm{F}}_j(\mathbf{x}) \psi_j(\boldsymbol{\xi}), \tag{10}$$

with $u^{\mathrm{F}}_j(\mathbf{x}) = \int_\Xi u(\mathbf{x}, \boldsymbol{\xi}) \psi_j(\boldsymbol{\xi}) w(\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi}$.

The choice of the $L^\infty(\Xi; H^1(\Omega))$-norm in (9) to define the error is not arbitrary. If we take a closer look at Proposition 3.1 and choose $d(u(\cdot, \boldsymbol{\xi}), u^{\mathrm{F}}(\cdot, \boldsymbol{\xi})) = \|u(\cdot, \boldsymbol{\xi}) - u^{\mathrm{F}}(\cdot, \boldsymbol{\xi})\|_{H^1}$, we have

$$d_\infty(\widetilde{u}, \widetilde{v}) \leqslant \|u - u^{\mathrm{F}}\|_{L^\infty(\Xi; H^1(\Omega))},$$

because $\|u - u^{\mathrm{F}}\|_{L^\infty(\Xi; H^1(\Omega))} = \sup_{\boldsymbol{\xi} \in \Xi} \|u(\cdot, \boldsymbol{\xi}) - u^{\mathrm{F}}(\cdot, \boldsymbol{\xi})\|_{H^1}$. Hence, we can say that convergence of u^{F} to u in the $L^\infty(\Xi; H^1(\Omega))$-norm implies convergence of $\widetilde{u}^{\mathrm{F}}$ to $\widetilde{u}$ in the fuzzy sense and also that this convergence will be of the same rate.

Obviously, we do not have the exact solution u and cannot construct the exact truncated Fourier series u^{F} directly. It is however approximated by the Galerkin projection u^{g}, which is defined as the solution of the finite-dimensional counterpart of Weak Formulation 1:

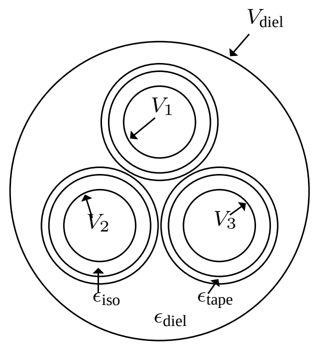
Weak Formulation 2. *Find*

$$u^g(\mathbf{x}, \boldsymbol{\xi}) = \sum_{i=1}^{N_\varphi} \sum_{j=1}^{N_\psi} u^g_{ij} \varphi_i(\mathbf{x}) \psi_j(\boldsymbol{\xi}) \in \Phi_{N_\varphi} \otimes \Psi_{N_\psi} \subset H^1_0(\Omega) \otimes L^2_w(\Xi),$$

such that

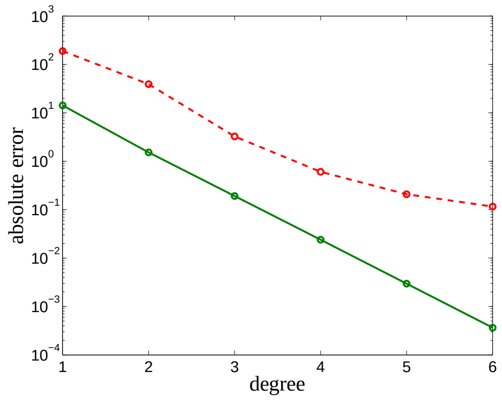
$$\int_\Xi \int_\Omega a \nabla_x u^g \cdot \nabla_x v \; w(\boldsymbol{\xi}) \mathrm{d}\mathbf{x} \mathrm{d}\boldsymbol{\xi} = \int_\Xi \int_\Omega f v \; w(\boldsymbol{\xi}) \mathrm{d}\mathbf{x} \mathrm{d}\boldsymbol{\xi}, \quad \forall v \in \Phi_{N_\varphi} \otimes \Psi_{N_\psi}.$$

Apart from the finite-dimensional space $\Psi_{N_\psi} \equiv \mathrm{span}\{\psi_1(\boldsymbol{\xi}), \psi_2(\boldsymbol{\xi}), \ldots, \psi_{N_\psi}(\boldsymbol{\xi})\}$, the space generated by the set of orthonormal polynomials and which discretizes the uncertainty domain, we also introduced $\Phi_{N_\varphi} \equiv \mathrm{span}\{\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots, \phi_{N_\psi}(\mathbf{x})\}$. It discretizes the spatial domain, and typically is chosen to be a finite element discretization. The Galerkin formulation possibly results in a very large system of algebraic equations in the unknowns u^{g}_{ij}. Luckily, it has a lot of structure and very efficient solvers are known. We refer the interested reader to [17].



boundary conditions (V)	
V_{diel}	0
V_1	-208.45
V_2	20.05
V_3	188.41
electric permittivity	
$\widetilde{\epsilon}_{\text{diel}}$	$\text{triag}(0.7, 1, 1.3) \cdot \epsilon_0$
$\widetilde{\epsilon}_{\text{tape}}$	$\text{triag}(4.8, 6, 7.2) \cdot \epsilon_0$
$\widetilde{\epsilon}_{\text{iso}}$	$\text{triag}(21.6, 24, 26.4) \cdot \epsilon_0$

Figure 1: Model of the 3-phase cable. The constant $\epsilon_0 = 8.85 \cdot 10^{-12} \frac{\text{F}}{\text{m}}$ is the vacuum permittivity. The operator $\text{triag}(a, b, c)$ represents a triangular fuzzy number with support $[a, c]$ and top b.



		setup time (s)	eval time (s)
PDE		0.3577	0.0487
Kriging	$D = 1$	0.6073	0.0008
	$D = 3$	1.6331	0.0010
	$D = 5$	4.4919	0.0017
Poly	$D = 1$	0.4998	0.0005
	$D = 3$	1.2908	0.0006
	$D = 5$	3.2071	0.0013

Figure 2: Absolute error of the polynomial response surface in full line, the Kriging surface in dashed line. Computation time of the setup and of one function evaluation are shown in the table.

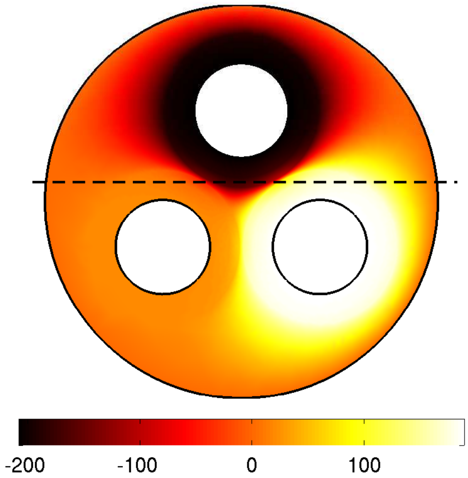
5 Numerical Experiment

We compare convergence and accuracy of response surfaces constructed by the polynomial approach and by a standard Kriging approach. The test problem is a simplified model of an electrical 3-phase cable that runs underground and carries $230V$. This model is taken from [16], where it is solved in the stochastic sense. Because the alternating current is relatively slowly varying, we can approximate a snapshot of the electrical potential in time by the static elliptic PDE (1), where the unknown u represents the voltage. Figure 1 shows a drawing of the model with Dirichlet boundary conditions on the outer shell and inner cables. The value of the voltages on the cables, as well as the fuzzy permittivity of the different insulating materials in the cable are shown in the table next to it.

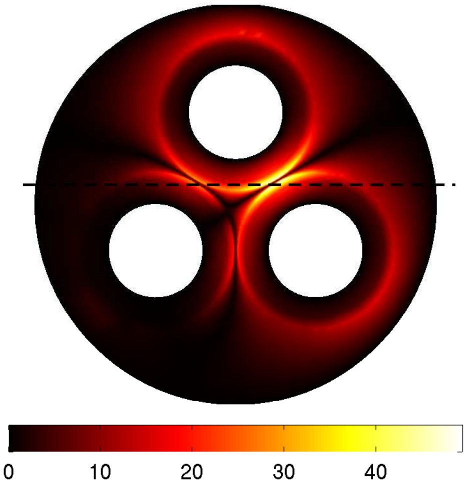
The set of basis functions Ψ_{N_ψ} that is used to construct the polynomial response surface contains the Chebyshev polynomials $\psi_j(\boldsymbol{\xi}) = \psi_{j1}(\xi_1)\psi_{j2}(\xi_2) \ldots \psi_{jN_\xi}(\xi_{N_\xi})$ of total degree smaller than or equal to a certain number D. This set is known as the set of Total Degree (TD) [4]. It has $\frac{(N_\xi+D)!}{N_\xi!D!}$ basis functions. To make a fair comparison, the Kriging surface is constructed with an equivalent number of samples, which are chosen by a Latin Hypercube sampling of the uncertainty domain Ξ. The spatial domain is discretized by a piecewise linear finite element space Φ_{N_φ} on a triangular mesh, with 19979 degrees of freedom.

Figure 2 shows the error of both response surfaces in the $L^\infty(\Xi; H^1(\Omega))$-norm as a function of degree D. Timings for setup and a single function evaluation of the PDE and both surfaces of different degree D are listed in the table next to it. The simulations were performed single-threadedly in Matlab R2009b on a computer equipped with an Intel Xeon X5550 of 2 x 2.66 GHz (8 cores) and 32GB of RAM.

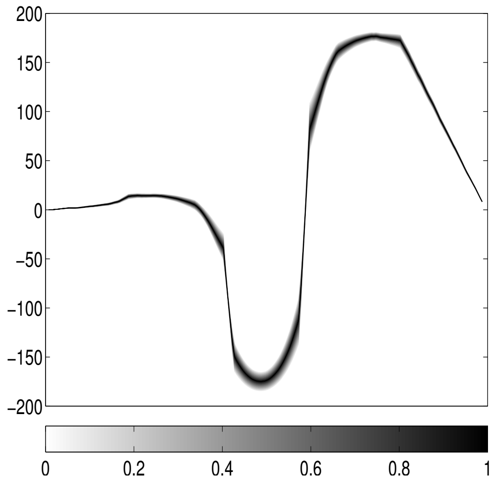
It is clear that convergence of the polynomial response surface constructed by a Galerkin approach is more regular than the unpredictable convergence of the Kriging surface. It is also more accurate for the same



(a) top



(b) diameter of support



(c) cross section

Figure 3: Figures 3a and 3b show the top of the fuzzy voltage field $\widetilde{u}(\cdot)$ and the diameter of the support of this field. Figure 3c shows the cross-section along the dashed line, with the gray-scale colors representing the membership value $\mu_{\widetilde{u}(\mathbf{x})}$. All the other units are in volt.

number of degrees of freedom. Looking at the timing results, we see that setup and evaluation time of both methods are of the same order. The PDE setup time, which consists of construction of the finite element matrices, is considerably lower than the setup time for the response surfaces. When many function evaluations are needed, like in fuzzy calculations, the response surface approach can however outperform the direct PDE method.

Finally, we also illustrate the actual solution of the fuzzy PDE, i.e., the resulting fuzzy electrical field of the problem. Figure 3 shows the top $[\widetilde{u}(\cdot)]_1$, the diameter of the support $\mathrm{diam}([\widetilde{u}(\cdot)]_0)$ and a cross-section along the dashed line. This fuzzy solution was calculated by the Transformation Method [8] with an additional Latin Hypercube sampling of the uncertainty domain Ξ.

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